THE UNIVERSITY OF ALBERTA

MAPPINGS OF PROXIMITY SPACES

by

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ABSTRACT

In a lecture at Humboldt University in 1959, M. Katetov suggested the desirability of studying mappings of proximity spaces. Quotients and other maps of proximity spaces have since been investigated by Katetov, Poljakov, Isbell, Dowker, Stone, and Nachman. This thesis continues the study begun by these authors.

Chapter I contains a history of the literature of proximity spaces and a brief introduction to the subject.

Chapter II is concerned with proximity quotients. A new explicit characterization of the quotient proximity is given. This characterization is used to find necessary and sufficient conditions on a proximity space for every proximity quotient map on this space to be a topological quotient map. It is shown that a separated proximity space X is compact iff every p-map on X with separated range is a proximity quotient map. Other mapping characterizations are obtained.

Much current research in general topology has been directed towards finding conditions under which the product of topological quotient maps is a quotient map. The main result of Chapter III is that the finite product of proximity quotient maps, each of which has a separated domain and range, is a proximity quotient map. This theorem is used to prove that if $f : X_1 \to Y_1$ and $g : X_2 \to Y_2$ are topological quotient maps between $T_{3\frac{1}{2}}$ spaces, then if $X_1 \times X_2$ is pseudocompact, $f \times g$ is a quotient map. Examples are given. It is also shown that the regular p-map

image of a semi-metrizable proximity space is semi-metrizable.

A metrization theorem for proximity spaces analogous to a topological metrization theorem of Morita is proved in Chapter IV. One consequence of this result is that if Y has the elementary proximity and is the image of a metrizable proximity space under a closed p-map, then Y is metrizable.

Products of proximity spaces are considered in Chapter V. A product of proximity spaces is defined which gives the elementary proximity on the product.

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CHAPTER I

INTRODUCTION TO PROXIMITY SPACES

1. History and Introduction.

A topology on a set X can be determined completely by a closure operator on the set $\mathcal{P}(X)$ of all subsets of X. In this sense, topology is an axiomatization of a point being "near" a set; that is, being in the closure of a set. It seems natural to axiomatize the concept of two sets being near. A proximity on a set is such an axiomatization.

The historical motivation for the introduction of proximity spaces has been somewhat different. There are at least two recurring notions of nearness in topology. The first, topological separation, was axiomatized in 1941 by A.D. Wallace [47]. Ten years later, E. Efremovich [9] considered two subsets of a metric space to be near whenever the distance between them was zero. It was this second concept of nearness that Efremovich axiomatized. and called proximity.

Much of the early work in proximity spaces was done by Y. Smirnov [43] and [44]. It was his theorem on the one-to-one correspondence between the compactifications of a completely regular space X and the compatible proximities on X which gave impetus to the study of these structures. Smirnov was also the first to explore the relationship between proximities and uniformities. Gal [14] and Alfsen and Fenstad [2] continued Smirnov's investigation and independently showed that there is a one-to-one correspondence between proximities and totally bounded uniformities.

Some of the recent areas of development in proximity spaces have been the generalizations due to Leader [22], Lodato [25], Pervin [36], and Harris [18]; the use of proximities to solve topological questions by Gagrat and Naimpally [11], [12], and [13]; lattices of proximities as studied by Dooher and Thron [7]; and mappings of proximity spaces, which will be our main concern here.

The study of mappings of proximity spaces was initiated by Katetov [20] in 1959. Although there has been some investigation of mappings by various authors, few of the characterization problems of current interest in general topology have been studied for proximity spaces.

In his survey paper "Mappings and Spaces" [3], A.V. Archangel'skii in 1965 presented a uniform approach to problems of the mutual classification of mappings and spaces. Let $\mathcal{F}$ and $\mathcal{G}$ be classes of maps and A and B classes of spaces. Archargel'skii isolated the following three problems:

(1) When is a space from class B an $\mathcal{F}$ - image of a space from class A .

(2) Characterize $A\mathcal{F}$: the class of spaces which are $\mathcal{F}$ - images of spaces of class A .

(3) Characterize $\mathcal{F}(A,B) \cap \mathcal{G}$: the intersection of $\mathcal{G}$ with the class of all maps with domain in class A and range in class B .

There are other general problems in mappings and spaces which have since been singled out for attention. MacDonald and Willard in [27] were interested in the following problem:

(4) Characterize the class of all spaces whose every $\mathcal{F}$ - image lies in $\mathcal{B}$.

Other similar problems are:

(5) Characterize the class of spaces X such that every map in $\mathcal{F}$ with domain X is in $\mathcal{G}$.

(6) Characterize the class of spaces Y such that every map in $\mathcal{F}$ with range Y is in $\mathcal{G}$.

These last two have not been isolated as such, but problems of type (5) have been considered by Dickman and Zane [6] and Willard [48], while Siwiec [42] and Lee [24] have solved problem (6) for various classes of maps and spaces.

Our interest in this thesis has been to answer some of these general classification problems for specific classes of maps on proximity spaces.

2. Basic Definitions.

A brief introduction to proximity spaces will be given in this section. For a more detailed discussion, the reader is referred to the text General Topology by S. Willard [49] and the monograph Proximity Spaces by S.A. Naimpally and B.D. Warrack [34]. Our development in this section will somewhat follow these two sources. All unproved propositions appear there.

2.1 Definition. A *proximity space* is a pair (X,δ) , where X is a set and δ is a binary relation on the collection $\mathcal{P}(X)$ of all subsets of X

such that

(P1) if $A\delta B$, then $A \neq \phi$ and $B \neq \phi$,

(P2) $A\delta B$ iff $B\delta A$,

(P3) if $A \cap B \neq \phi$, then $A\delta B$,

(P4) $A\delta(B \cup C)$ iff $A\delta B$ or $A\delta C$,

(P5) if $A\not\delta B$, then there exist disjoint sets C and D such that $A\not\delta(X-C)$ and $B\not\delta(X-D)$,

where $A\not\delta B$ means it is not true that $A\delta B$. The proximity space (X,δ) is called <u>separated</u> if it also satisfies

(P6) $a\delta b$ iff $a = b$.

When no confusion can result, we shall speak of the proximity space X . The phrase $A\delta B$ is read "A is near B".

<u>2.2 Examples</u>. (a) In a metric space, define $A\delta B$ iff $d(A,B) = 0$. A proximity which is given by a metric is called <u>metrizable</u>.

(b) In a normal topological space, define $A\delta B$ iff $\overline{A} \cap \overline{B} \neq \phi$. This proximity is called the <u>elementary proximity</u>.

(c) If (X,μ) is a covering uniformity, let $A\delta B$ iff $St(A,U) \cap B \neq \phi$ for all $U \in \mu$. A proximity obtainable in this way is called <u>uniformizable</u>. It is known that every proximity is uniformizable; in fact, obtainable from a unique totally bounded uniformity.

(d) In a completely regular space X , define $A\not\delta_0 B$ iff there is a continuous function f mapping X to the unit interval I such

that $f(A) = 0$ and $f(B) = 1$. This proximity is called the <u>fine proximity</u> on X . It is equivalent to the elementary proximity when X is normal.

(e) In a set X , define $A\delta B$ iff $A \neq \phi$ and $B \neq \phi$. This is the <u>trivial proximity</u>.

<u>2.3 Proposition</u>. If (X,δ) is a proximity space, then δ induces a completely regular topology on X such that $\overline{A} = \{x \mid x\delta A\}$. Conversely, if X is any completely regular space, the fine proximity on X induces the original topology.

<u>2.4 Remarks</u>. $\zeta(\delta)$ will denote the topology induced by δ . If (X,ζ) is a topological space, a proximity δ on X will be called <u>compatible</u> iff $\zeta(\delta) = \zeta$. Motivated by the fact that a set U is a neighborhood of a point $x \in X$ in $\zeta(\delta)$ iff $x\not\delta(X-U)$, we shall write $A \subset\subset B$ to mean $A\not\delta(X-B)$ and read "B is a δ-neighborhood (or p-neighborhood) of A". Then axiom (P5) in definition 2.1 can be rewritten as

(P5') if $A \subset\subset B$, there is some set E such that $A \subset\subset E \subset\subset B$.

A continuous function is one which satisfies $f(\overline{A}) \subseteq \overline{f(A)}$ for any $A \in P(X)$. Another way of expressing this is to say that f is continuous whenever x near A implies $f(x)$ near $f(A)$. Proximity maps on proximity spaces arise naturally as functions which take near sets to near sets. Formally, we have the following:

2.5 Definition. If f is a function between the proximity spaces (X,δ) and (Y,δ'), f is a p-map iff $f(A)\delta' f(B)$ whenever $A\delta B$.

Clearly, every p-map is continuous in the induced topologies on X and Y. If δ is the fine proximity the converse is also true.

2.6 Definition. A one-to-one, onto map f such that both f and f^{-1} are p-maps is a p-isomorphism.

We turn now to the construction of the Smirnov Compactification of a proximity space. In a completely regular space X, the Stone-Cech Compactification of X is just the set of all ultrafilters on X (with a suitable topology). It is well-known that an ultrafilter $\mathcal{F}$ can be characterized as a subset of $\mathcal{P}(X)$ such that

(a) if $A,B \in \mathcal{F}$, then $A \cap B \neq \phi$,

(b) if $A \cup B \in \mathcal{F}$, then $A \in \mathcal{F}$ or $B \in \mathcal{F}$, and

(c) if $A \cap B \neq \phi$ for all $B \in \mathcal{F}$, then $A \in \mathcal{F}$.

Analogously, S. Leader [21] in 1959 defined a cluster as follows:

2.7 Definition. A cluster π on a proximity space (X,δ) is a subset of $\mathcal{P}(X)$ such that

(a) if $A,B \in \pi$, then $A\delta B$,

(b) if $A \cup B \in \pi$, then $A \in \pi$ or $B \in \pi$,

(c) if $A\delta B$ for all $B \in \pi$, then $A \in \pi$.

Leader used clusters to construct the compactification due to Smirnov. Before stating the theorem, we shall need some definitions.

2.8 Definition. If δ and δ' are two proximities on a set X, δ is said to be finer than δ' iff $A\delta B$ implies $A\delta' B$. This is written $\delta' < \delta$. If $\delta' < \delta$ and $\delta < \delta'$, δ and δ' are called equivalent.

Given a completely regular space X, the fine proximity is the finest proximity compatible with X.

2.9 Definition. If K_1 and K_2 are compactifications of X we write $K_1 < K_2$ iff there exists a continuous $F : K_2 \to K_1$ such that $F \circ h_2 = h_1$, where h_1 and h_2 are embeddings of X into K_1 and K_2 respectively.

As is well-known, the Stone-Cech Compactification is the largest compactification under the ordering of definition 2.9. One consequence of part (e) of the next theorem, then, is that the Stone-Cech Compactification is the Smirnov Compactification of the fine proximity.

2.10 Theorem. (Smirnov-Leader). Let (X,δ) be a separated proximity space. Then:

(a) X is compact iff every cluster on X contains a point.

(b) The space X^* of all clusters on X (with a suitable topology) is a Hausdorff compactification of X.

(c) X^* is the unique compactification of X such that any p-map from X into a separated proximity space Y has a unique continuous extension $f^* : X^* \to Y^*$.

(d) $A \not\delta B$ iff $Cl_{X^*} A \cap Cl_{X^*} B = \phi$ iff there is a p-map $f : X \to I$ such that $f(A) = 0$ and $f(B) = 1$.

(e) There is a one-to-one order preserving correspondence between the Hausdorff compactifications of X and the compatible proximities on X .

CHAPTER II

QUOTIENTS OF PROXIMITY SPACES

3. Introduction.

In 1959 Katetov [20] introduced proximity quotient maps and suggested that mappings of proximity spaces be investigated. Quotients have since been studied by Isbell [19], Nachman [32] and Stone [46], while proximity open maps were defined and examined by Poljakov in [37] and [38]. Although there are characterizations of proximity quotient maps in the literature [32], the only explicit formulation of the quotient proximity the author knows of is due to A.H. Stone, whose work appears in the general topology text by Willard [49]. Our purpose in the present chapter is to provide another approach to proximity quotient maps; we will then use our new characterization to study mapping properties of proximity spaces.

In this chapter (X,δ) will always denote a (not necessarily separated) proximity space and δ_o will represent the fine proximity (2.2d) on X .

4. Characterization.

4.1 Definition. Let f be a function from a proximity space (X,δ) onto a set Y . The quotient proximity is the finest proximity on Y such that f is a p-map. When Y has the quotient proximity, f will be called a p-quotient map.

4.2 Theorem. (Stone [46]). The quotient proximity is given by: $C \ll D$ iff for each binary rational $s \in [0,1]$, there is some $C_s \subseteq Y$ such that $C_0 = C$, $C_1 = D$ and $s < t$ implies $f^{-1}(C_s) \ll f^{-1}(C_t)$.

The above characterization, although explicit, is difficult to work with for obvious reasons. In 4.3-4.5 we introduce a simpler approach and prove it works.

4.3 Definition. Let f be a function from a proximity space (X,δ) onto a set Y. Define $A \not\delta' B$ in Y iff there is a function $g : Y \to I$ such that $g(A) = 0$, $g(B) = 1$ and $g \circ f$ is a p-map.

4.4 Lemma. δ' is a proximity.

Proof. Clearly, the axioms (P1)-(P3) of definition 2.1 hold, and $A\delta' B$ or $A\delta' C$ easily implies $A\delta'(B \cup C)$. Let $A \not\delta' B$ and $A \not\delta' C$. We must show $A \not\delta' (B \cup C)$. Let g and h map Y to I such that $g(A) = 0$, $g(B) = 1$, $h(A) = 0$, $h(C) = 1$ and $g \circ f$ and $h \circ f$ are p-maps. Then $(g \cdot h)(A) = 0$, $(g \cdot h)(B \cup C) = 1$, and since the product of bounded real-valued p-maps is a p-map [20, Proposition 2.2], $(g \cdot h) \circ f = (g \circ f) \cdot (h \circ f)$ is thus a p-map. Therefore, $A \not\delta' (B \cup C)$, and axiom (P4) is satisfied.

To prove (P5), let $A \not\delta' B$. Then by definition, there is a function $g : Y \to I$ such that $g(A) = 0$, $g(B) = 1$ and $g \circ f$ is a p-map. Let $C = g^{-1}[0,\frac{1}{3})$ and $D = g^{-1}(\frac{2}{3},1]$. Then $C \cap D = \phi$ and it is easy to show that $A \not\delta' (X-C)$ and $B \not\delta' (X-D)$.

4.5 Theorem. δ' is the quotient proximity.

Proof. First, $f : (X,\delta) \to (Y,\delta')$ is a p-map. For if $A \not\delta' B$, there is some $h : Y \to I$ such that $h(A) = 0$, $h(B) = 1$ and $h \circ f$ is a p-map. It follows that $f^{-1}h^{-1}(0) \not\delta f^{-1}h^{-1}(1)$ and therefore $f^{-1}(A) \not\delta f^{-1}(B)$.

Now, if δ^* is another proximity on Y such that $f : (X,\delta) \to (Y,\delta^*)$ is a p-map, then $A \not\delta^* B$ implies there is a p-map $h : (Y,\delta^*) \to I$ such that $h(A) = 0$ and $h(B) = 1$. But then, $A \not\delta' B$ and hence δ' is the finest proximity on Y for which f is a p-map.

4.6 Corollary. A function g satisfying definition 4.3 is continuous relative to the quotient topology on Y .

Proof. Let U be open in I . Then $V = f^{-1}g^{-1}(U)$ is open in X since every p-map is continuous. Since $V = f^{-1}f(V)$, it follows that $f(V) = g^{-1}(U)$ is open in the quotient topology on Y .

Notation. In the sequel δ/f will denote the quotient proximity on Y induced by f and (X,δ) . When no confusion can result, this will be abbreviated δ' .

The next theorem is proved quite easily from theorem 4.5. The result has also been established by Nachman [32], who used the techniques of uniform spaces. We shall need to make use of it in chapter III, and so state it here without proof.

4.7 Theorem. If (X,δ) and (Y,δ') are proximity spaces and $f : X \to Y$ is onto, then f is a p-quotient map iff

(1) f is a p-map and

(2) $g \circ f$ is a p-map whenever $g : Y \to (Z,\gamma)$ iff g is a p-map.

4.8 Corollary. If Y has the quotient proximity induced by $f : (X,\delta) \to Y$, then Y is p-isomorphic to the natural decomposition space X' (that is, the space whose "points" are the sets $f^{-1}(y)$ for every $y \in Y$ and whose proximity is the quotient proximity induced by the function $\psi : X \to X'$ which takes $x \in X$ to the set in which it is contained).

4.9 Theorem. Let f be a function mapping the set X onto the set Y and let δ and γ be two proximities on X with $\gamma < \delta$. Then $\gamma/f < \delta/f$.

Proof. We note that $f : (X,\delta) \to (Y,\delta/f)$ and $f : (X,\gamma) \to (Y,\gamma/f)$ are p-maps and we claim $f : (X,\delta) \to (Y,\gamma/f)$ is a p-map. To prove this, let $A\delta B$ and assume $f(A)$ and $f(B)$ are γ/f - separated. Then there is some $g : Y \to I$ such that $g(f(A)) = 0$, $g(f(B)) = 1$ and $g \circ f$ is a p-map relative to γ. Since $\gamma < \delta$, $g \circ f$ must also be a p-map relative to δ. But then, by definition of the quotient proximity δ/f, $f(A)$ and $f(B)$ are δ/f - separated, which cannot happen since $A\delta B$ and $f : (X,\delta) \to (Y,\gamma/f)$ is a p-map. Thus, $f : (X,\delta) \to (Y,\gamma/f)$ is a p-map and so by the definition of δ/f, $\gamma/f < \delta/f$.

Although the quotient proximity is usually given neither by $A\delta'B$ iff $f^{-1}(A)\ \delta\ f^{-1}(B)$ nor by the elementary proximity, there is at least one

case in which it is given by a "nice" combination of the two.

4.10 Theorem. Let X_o be a closed subset of a proximity space (X,δ) and identify X_o to a point y_o. Then $A\delta'B$ iff either $y_o \in Cl_{\zeta(\delta')}A \cap Cl_{\zeta(\delta')}B$ or $f^{-1}(A)\ \delta\ f^{-1}(B)$, where f is the natural map.

Proof. Clearly, $A\delta'B$ whenever either property holds. Now assume both $f^{-1}(A)\ \not\delta\ f^{-1}(B)$ and $y_o \notin Cl_{\zeta(\delta')}A \cap Cl_{\zeta(\delta')}B$, say $y_o \notin Cl_{\zeta(\delta')}A$. Then $f^{-1}(A)\ \not\delta\ X_o$. Let $g : X \to I$ be a p-map such that $g(f^{-1}(A)) = 0$ and $g(f^{-1}(B) \cup X_o) = 1$. Define $h : Y \to I$ as follows:

$$h(x) = \begin{cases} g(f^{-1}(x)) & , \quad x \neq y_o \\ 1 & , \quad x = y_o \end{cases} .$$

Then $h(A) = 0$ and $h(B) = 1$, for $x \notin X_o$, $(h\circ f)(x) = g(f^{-1}f(x)) = g(x)$, and for $x \in X_o$, $(h\circ f)(x) = h(y_o) = 1 = g(x)$. So $h\circ f = g$ is a p-map. Thus, by the definition of the quotient proximity, $A\not\delta'B$. The result follows.

4.11 Corollary. If $X_1,\cdots,X_n$ are pairwise separated subsets of X, and Y is formed by identifying each X_i to a point y_i, then $A\delta'B$ iff either $y_i \in cl_{\zeta(\delta')}A \cap cl_{\zeta(\delta')}B$ for some y_i, $i = 1,\cdots,n$ or $f^{-1}(A)\ \delta\ f^{-1}(B)$, where f is the natural map.

4.12 Example. A proximity space with a countable, locally finite collection of closed subsets, and a quotient space formed by identifying each member of

this collection to a point, with the quotient proximity not given as a "nice" combination of the elementary proximity and $A\delta' B$ iff $f^{-1}(A)\ \delta\ f^{-1}(B)$.

Let $X = R\times R$ with the metric proximity, $A = R\times\{1\}$, and $B = R\times\{0\}$. For each integer $n > 3$, let $F_n = \{n\}\times[\frac{1}{n}, 1 - \frac{1}{n}]$. Now, identify each F_i to a point y_i and let Y be the resulting space and f the natural map. We claim that $f(A)\delta' f(B)$. If not, there is some U such that $f(B) \subset\subset U$ and $U \not\delta' f(A)$. If U contains infinitely many y_is , then $f^{-1}(U)$ contains infinitely many F_i , and so is near A . It follows that $U\ \delta'\ f(A)$ - a contradiction. On the other hand, if U contains only finitely many y_i , we may assume it contains no y_i , so $f^{-1}(U)$ intersects no F_i and hence $(X - f^{-1}(U))\ \delta\ B$. But then it follows that $f(X - f^{-1}(U))\ \delta\ f(B)$ and since $f(X - f^{-1}(U)) = Y - U$, we again have a contradiction.

5. Proximity Quotients vs. Topological Quotients.

In this section we consider the question of when a proximity quotient map on a proximity space is a topological quotient map.

5.1 Example. A p-quotient map f on the real line with $f^{-1}(y)$ finite for all y in the quotient, but f not a topological quotient map.

Let X be the non-negative real line with the usual proximity, identify n and $\frac{1}{n}$ for each positive integer n , and let Y be the resulting set. Consider $U = [0,\frac{1}{2}] \cup \bigcup_{n>2} \{(n - \frac{1}{n}, n + \frac{1}{n})\}$. Since $U = f^{-1}f(U)$,

where f is the natural map, $f(U)$ is open in the quotient topology on Y. However, if $f(0) \subset\subset f(U)$, there must be some V such that $f(0) \subset\subset V \subset\subset f(U)$. Now, $\langle f(n) \rangle$ converges to $f(0)$, so there is some $N > 2$ with $f(n) \in V$ for all $n \geq N$. Thus, $\{f(n)\}_{n>N} \not\delta' Y - f(U)$. But clearly, $\{n\}_{n>N} \delta (X-U)$, so that $\{f(n)\}_{n>N} \delta' Y - f(U)$. Therefore, $f(U)$ is not a $\zeta(\delta')$ - neighborhood of $f(0)$ and $\zeta(\delta')$ is not the quotient topology even though $f^{-1}(y)$ is finite for all $y \in Y$.

It is well known that a one-to-one topological quotient map is a homeomorphism. The next theorem demonstrates that the analogue for proximity spaces is also true. Note that part (1) exhibits one-to-one p-quotients in a form one might expect all p-quotients to take. That this would not be a viable approach to p-quotients in general was observed in 4.10-4.12. Also, part (3) provides us with a first approximation to the main question of this section.

<u>5.2 Theorem.</u> Let $f : (X,\delta) \to (Y,\delta')$ be a one-to-one p-quotient map. Then:

(1) $A\delta' B$ iff $f^{-1}(A) \ \delta \ f^{-1}(B)$,

(2) f is a p-isomorphism, and

(3) the topology induced by δ' is the quotient topology.

<u>Proof.</u> (1) Since f is a p-map, $f^{-1}(A) \ \delta \ f^{-1}(B)$ implies $A\delta' B$. So, let $f^{-1}(A) \not\delta f^{-1}(B)$ and let $g : X \to I$ be a p-map such that $g(f^{-1}(A)) = 0$ and $g(f^{-1}(B)) = 1$. If $h : Y \to I$ is defined by $h(y) = g(f^{-1}(y))$, then $h \circ f = g$ is a p-map. Hence, $A \not\delta' B$. (2) and (3) follow easily from (1).

If we restrict the proximity on X , we obtain the next partial solution to the problem.

5.3 Theorem. If δ_o is the fine proximity on X and $f : (X,\delta_o) \to (Y,\delta')$ is a p-quotient map, then $\zeta(\delta')$ is the quotient topology iff the quotient topology is completely regular.

Proof. Let the quotient topology, α , be completely regular and let δ^* be any proximity compatible with α . Since δ_o is the fine proximity on X , $f : (X,\delta_o) \to (Y,\delta^*)$ is a p-map by the remark following definition 2.5. The quotient proximity is the finest proximity on Y for which f is a p-map, so $\delta^* < \delta'$, and hence $\alpha = \zeta(\delta^*) \subseteq \zeta(\delta')$. Now, since every p-map is continuous and the quotient topology is the finest topology on Y for which f is continuous, it must also be true that $\zeta(\delta') \subseteq \alpha$. Therefore, $\zeta(\delta') = \alpha$.

Necessity is obvious.

While it is not true that if $\zeta(\delta')$ is the quotient topology then δ is the fine proximity on X , the following holds:

5.4 Theorem. Let (X,δ) be a proximity space. Then every proximity quotient of X generates the quotient topology iff

(1) δ is the fine proximity and

(2) every (topological) quotient is completely regular.

Proof. Sufficiency follows from theorem 5.3. For the necessity, assume every proximity quotient generates the quotient topology. If δ is not the

fine proximity δ_o , there are two sets A and B such that $A\delta B$ but $A\not\delta_o B$. That is, A and B are functionally separated. It follows that there is an open set U such that $\overline{A} \subseteq U$ and $U \cap B = \phi$. Let Y be the set formed by identifying $\overline{A}$ to a point, and give Y the quotient proximity. Then, if f is the natural map, f(U) is an open neighborhood of $f(\overline{A})$ in the quotient topology. But $\overline{A}\delta B$ implies $f(\overline{A})\delta' f(B)$, so $f(\overline{A}) \in Cl_{\zeta(\delta')} f(B)$. Clearly, f(U) cannot be a $\zeta(\delta')$ - neighborhood of $f(\overline{A})$, a contradiction. Condition (2) above easily holds.

Remarks. Exactly the same proof will show that every separated quotient generates the quotient topology iff (1') δ is the fine proximity and (2') every T_2 quotient is completely regular. The problem of characterizing the topological spaces X whose every quotient is completely regular seems to be difficult; for related work, see MacDonald and Willard [27]. Note that theorem 5.4 is a solution to the general problem (5) of section 1 for the class $\mathcal{F}$ of p-quotient maps with separated range and the class $\mathcal{G}$ of topological quotient maps.

6. Mapping Properties.

Our purpose here is to give solutions to problems (5) and (6) of section 1 for various maps on proximity spaces.

6.1 Theorem. Let (X,δ) be a separated proximity space. Then every p-map on X with separated range is a p-quotient map iff (X,δ) is compact.

Proof. Assume (X,δ) is not compact. Then by theorem 2.10(a) there is a cluster π without a point. Let p be any point in X and define $\pi' = \pi \cup \pi_p$, where π_p is the cluster of all A such that $A\delta p$. Finally, define

$$A\delta' B \text{ iff either } A\delta B \text{ or both } A \text{ and } B \in \pi' .$$

We claim that δ' is a proximity on X . Axioms (P1)-(P3) of definition 2.1 are easily verified. To prove (P4) , observe that π' , as the union of two clusters, inherits the following property of clusters:

(i) $$B \cup C \in \pi' \quad \text{iff} \quad B \in \pi' \quad \text{or} \quad C \in \pi' .$$

Now, let $A\delta'(B\cup C)$. If $A\delta(B\cup C)$, we are done, so assume $A\not\delta(B\cup C)$. Then we must have $A \in \pi'$ and $(B\cup C) \in \pi'$. It follows from (i) that $A\delta' B$ or $A\delta' C$. The reverse implication in (P4) also follows from (i) .

To prove (P5) , let $A\not\delta' B$. Then either $A \notin \pi'$ or $B \notin \pi'$. By symmetry we need only consider the case where $A \notin \pi'$. Then, $A\not\delta B$, $A\not\delta p$, and since $A \notin \pi$, $A\not\delta P$ for some $P \in \pi$. Thus, there exist disjoint sets U and V such that $A\not\delta(X-U)$ and $(B\cup P\cup\{p\})\not\delta(X-V)$. But $(X-V) \notin \pi$ since $(X-V)\not\delta P$ and $P \in \pi$, and $(X-V) \notin \pi_p$ since $(X-V)\not\delta p$, so $(X-V) \notin \pi'$. Since A and $(X-V)$ are not in π' , $A\not\delta'(X-U)$ and $(B\cup P\cup\{p\})\not\delta'(X-V)$, and (P5) is satisfied.

Note that δ' is easily separated and $\delta' < \delta$. It follows from our assumption that $i : (X,\delta) \to (X,\delta')$ is a one-to-one p-quotient map, and so by theorem 5.2, $\delta = \delta'$. This contradicts the definition of δ' . There-

fore (X,δ) is compact.

Conversely, assume (X,δ) is compact and let f be a p-map from X onto a separated proximity space (Y,δ^*). Then the quotient topology is $T_{3\frac{1}{2}}$ and δ is the fine proximity on X. It follows from theorem 5.3 that $\zeta(\delta')$ is the quotient topology α. Since $f : (X,\zeta(\delta)) \to (Y,\zeta(\delta^*))$ is a continuous function on a compact set, it must also be true that $\zeta(\delta^*) = \alpha$. But since (Y,α) is compact and Hausdorff, it has a unique compatible proximity (theorem 2.10); hence δ^* is the quotient proximity and so f is p-quotient.

<u>Remarks</u>. If it is not required that the range be separated in theorem 6.1, then δ must be trivial. That is, if every p-map on (X,δ) is a p-quotient map, then the identity $i : (X,\delta) \to (X,\delta^*)$, where δ^* is the trivial proximity of example 2.2(e), must be a one-to-one p-quotient map. Hence, δ would be equivalent to δ^*.

It is well known that a $T_{3\frac{1}{2}}$ space X is locally compact iff it has a minimal compatible proximity. If we consider a separated proximity δ on a set X to be <u>minimal separated</u> whenever $\delta^* < \delta$ and δ^* separated imply $\delta = \delta^*$, then we have the following.

<u>6.2 Corollary</u>. Let (X,δ) be a separated proximity space. Then (X,δ) is compact iff δ is minimal separated.

Our next result is a solution to problem (6) of section 1 for the classes F of all p-maps that are quotient maps, and G of all p-quotient maps.

6.3 Theorem. Let Y be any completely regular topological space and δ^* any compatible proximity. Then δ^* is the fine proximity on Y iff for all proximity spaces (X,δ) and all p-maps $f : (X,\delta) \to (Y,\delta^*)$ onto Y which are topological quotient maps, f is a p-quotient map.

Proof. Let $\delta^* = \delta_o$, the fine proximity on Y, and let $f : (X,\delta) \to (Y,\delta_o)$ be a p-map onto Y such that $\zeta(\delta_o)$ is the quotient topology α. Then by definition of the quotient proximity, $\delta_o < \delta'$. If $A \not\delta' B$, there is some $g : Y \to I$ such that $g(A) = 0$, $g(B) = 1$, $g \circ f$ is a p-map and g is continuous relative to α. But then $A \not\delta_o B$, so $\delta_o = \delta'$ and f is a p-quotient map.

For the converse, consider $i : (Y,\delta_o) \to (Y,\delta^*)$. This is a one-to-one p-map and a topological quotient map, and so by our assumption a p-quotient map. It follows from theorem 5.2 that $\delta_o = \delta^*$.

6.4 Corollary. Let (X,δ) be a proximity space and $(Y,\delta/f)$ a p-quotient of X such that the quotient topology is completely regular. Then δ/f is the fine proximity on the quotient topology iff $\delta/f = \delta_o/f$, where δ_o is the fine proximity on X.

Proof. If $\delta/f = \delta_o/f$ and the quotient topology is completely regular, then by theorem 5.3, $\zeta(\delta_o/f)$ is the quotient topology. But if $\delta^* > \delta_o/f$ and $\zeta(\delta^*) = \zeta(\delta_o/f)$, then $f : (X,\delta_o) \to (Y,\delta^*)$ is continuous and, since δ_o is the fine proximity, a p-map. So by definition we have $\delta^* < \delta_o/f$. It is then clear that $\delta^* = \delta_o/f$, so that δ_o/f is the finest proximity on the topological quotient.

Conversely, if δ/f is the fine proximity on the topological quotient, consider $f \circ i : (X,\delta_o) \to (X,\delta) \to (Y,\delta/f)$. Clearly $f \circ i = f$ is a p-map, and since $\zeta(\delta/f)$ is the quotient topology, it is also a (topological) quotient map. Then by theorem 6.3, $f \circ i$ is a p-quotient map; that is, $\delta/f = \delta_o/f \circ i = \delta_o/f$.

Remark. Note that $\delta/f = \delta_o/f$ does not necessarily imply that δ is the fine proximity on X . For if X is the real line with the metric proximity and

$$f(x) = \begin{cases} 0 & , \quad x \leq 0 \\ x & , \quad 0 < x < 1 \\ 1 & \quad\quad 1 \leq x \end{cases}$$

then the quotient space I has the fine proximity although X does not.

We turn now to characterizations of p-open maps.

6.5 Definition. (Poljakov [37] and [38]). A p-map $f : (X,\delta) \to (Y,\delta^*)$ is p-open iff $A \subset\subset B$ implies $f(A) \subset\subset f(B)$.

It is not hard to see that p-open onto maps are both p-quotient and open. The proof of the next theorem is routine and thus omitted.

6.6 Theorem. A p-map $f(X,\delta) \to (Y,\delta^*)$ onto Y is p-open iff $A \ \delta \ f^{-1}(B)$ whenever $f(A) \ \delta^* \ B$.

6.7 Theorem. Let (X,δ) be a proximity space such that every p-quotient map on X is p-open. Then every p-quotient of X generates the quotient

topology. Consequently, δ is the fine proximity on X .

Proof. If X has a p-quotient $(Y,\delta/f)$ which does not generate the quotient topology, f cannot be open, hence not p-open, a contradiction. The "consequence" follows from theorem 5.4.

Remark. There do not seem to be reasonable sufficient conditions on a proximity space X such that every p-quotient map on X is p-open. For example, if $X = [0,2]$ and $[\frac{1}{2},\frac{3}{2}]$ is identified to a point, then X is a compact metric space but the natural map f is a p-quotient map which is not open, and so not p-open. (The set $(\frac{1}{2},\frac{3}{2})$ is open in X , but $f(\frac{1}{2},\frac{3}{2})$ is a point, and so not open in the quotient.) A related topological question is to characterize the topological spaces X such that every quotient map on X is open. Again, the problem seems difficult. See McDonald and Willard [27] for similar problems.

CHAPTER III

REGULAR MAPS AND PRODUCTS OF P-QUOTIENT MAPS

7. Introduction.

Much current research in general topology has been concerned with generalizations of topological quotient maps. Two of these generalizations are hereditarily quotient maps, which have been considered by Archangel'skii [4] and Michael [28], and the bi-quotient maps of Michael [28] and Hajek [17]. These two mappings have analogues in proximity spaces which, in general, preserve more structure than their topological counterparts.

8. Regular Maps.

Poljakov in [38] introduced regular maps and asked if the regular image of a metrizable proximity space is metrizable. In [39] he showed that a proximity space can be "determined by sequences" iff it is the regular image of the disjoint union of metrizable proximity spaces. However, since the disjoint union of metrizable proximity spaces might not be a metrizable proximity space (although the induced topology is, of course, metrizable), this did not answer the original question. The purpose of this section is to give a partial solution to the problem. We begin with the definitions and a characterization due to Poljakov.

8.1 Definition. A p-map $f : (X,\delta) \to (Y,\delta')$ is regular iff $A\delta' B$ implies $f^{-1}(A)\ \delta\ f^{-1}(B)$.

8.2 Theorem. Let $f : (X,\delta) \to (Y,\delta')$ be a p-map. Then the following are equivalent:

(1) f is regular

(2) $f^{-1}(A) \ll U \Rightarrow A \ll f(U)$

(3) $f\big|_{f^{-1}(S)} : f^{-1}(S) \to S$ is p-quotient for all $S \subseteq Y$.

8.3 Remarks. (a) The equivalence of (2) and (3) above is similar to Archangel'skii's result in [4] which states that a map is hereditarily quotient iff whenever U is a neighborhood of $f^{-1}(x)$, $f(U)$ is a neighborhood of x .

(b) Theorem 6.6 implies that every p-open map is regular and (3) above shows that regular maps are p-quotient. Poljakov [38] gives examples to show that the reverse implications are not true in general.

Our intention is to show that the regular image of a semi-metrizable proximity space is semi-metrizable. First, some more definitions.

8.4 Definition. A semi-metric on a set X is a real-valued function d on $X \times X$ such that for all x and y in X ,

(a) $d(x,y) = d(y,x) \geq 0$, and

(b) $d(x,y) = 0$ iff $x = y$.

8.5 Definition. A proximity space (X,δ) is semi-metrizable iff there is a semi-metric d on X such that $A\delta B$ iff $d(A,B) = 0$.

We shall need a lemma which may be of some interest in itself. It is a proximity analogue of a topological semi-metrization theorem due independently to C.M. Pareek [35] and C.C. Alexander [1]. Gagrat and

Naimpally [13] have also recently considered the semi-metrization of proximity spaces. The emphasis of their research however, was to use proximities to obtain topological results.

8.6 Lemma. A separated proximity space (X,δ) is semi-metrizable iff there is a countable family $\{V_i\}_{i=1}^{\infty}$ of symmetric subsets of $X\times X$ satisfying:

(a) $\bigcap_{i=1}^{\infty} V_i = \Delta$ (the diagonal), and

(b) for each closed subset A of X, $\{V_i[A]\}_{i=1}^{\infty}$ forms a δ-neighborhood base for A ($A \subset\subset V_i[A]$ for all i and if $A \subset\subset B$, then $A \subset\subset V_N[A] \subset B$ for some N).

Proof. <= Assume $V_{i+1} \subset V_i$ and let

$$d(x,y) = 0 \quad \text{iff} \quad (x,y) \in V_i \quad \text{for all} \quad i$$

$$d(x,y) = 1 \quad \text{iff} \quad (x,y) \notin V_i \quad \text{for any} \quad i$$

$$d(x,y) = \frac{1}{i+1} \quad \text{iff} \quad (x,y) \in V_i - V_{i+1} .$$

Then d is a semi-metric. Now, let $A\not\delta B$, i.e. $A \subset\subset X-B$. By our assumption, $A \subset\subset V_N[A] \subset X-B$ for some N. For each pair $(a,b) \in A\times B$, it must be true that $d(a,b) \geq \frac{1}{N}$, since if $d(a_o,b_o) < \frac{1}{N}$ for some $(a_o,b_o) \in A\times B$, then $b_o \in V_N[A]$ - a contradiction. Therefore, $d(A,B) \geq \frac{1}{N} > 0$.

Conversely, if $d(A,B) = \epsilon > 0$, pick a positive integer N such that $\frac{1}{N} < \epsilon$. Then $V_N[A] \cap B = \phi$, so that $A \subset\subset V_N[A] \subset X-B$, and $A\not\delta B$.

=> Let (X,δ) be semi-metrizable with semi-metric d. Let $V_i = \{(x,y) \in X\times X \mid d(x,y) < \frac{1}{i}\}$. Clearly, $\{V_i\}_{i=1}^{\infty}$ has the required properties.

<u>8.7 Corollary</u>. A separated proximity space (X,δ) is semi-metrizable iff there is a countable family $\{U_i\}_{i=1}^{\infty}$ of covers of X such that

(a) $U_{i+1} < U_i$ and

(b) For each closed subset A of X, $\{St(A, U_i)\}_{i=1}^{\infty}$ forms a δ-neighborhood base for A.

<u>Proof</u>. If (X,δ) is semi-metrizable, let $\{V_i\}_{i=1}^{\infty}$ be the sequence that exists by the lemma. If $U_i = \bigcup_{x\in X} V_i[x]$, $\{U_i\}$ has the required properties.

Conversely, starting with a sequence $\{U_i\}$ of covers with the properties (a) and (b), let $V_i = \cup\{U\times U \mid U \in U_i\}$. Then the conditions of 8.6 are satisfied, so (X,δ) is semi-metrizable.

<u>8.8 Theorem</u>. Let f be a regular map from a semi-metrizable proximity space (X,δ) onto the separated proximity space (Y,δ'). Then (Y,δ') is semi-metrizable.

<u>Proof</u>. Since (X,δ) is semi-metrizable, there is a sequence $\{V_i\}_{i=1}^{\infty}$ of covers of X such that the star at any closed subset of X forms a δ-neighborhood base. Let A be a closed subset of Y. Clearly, $f^{-1}(A)$ is a closed subset of X and $f^{-1}(A) \subset\subset St(f^{-1}(A),V_i)$ for all i. Since f is regular, it follows from theorem 8.2 that

$A \subset\subset f(St(f^{-1}(A),V_i)) = St(A,f(V_i))$ for all i. Also, if $A \subset\subset B$ then $f^{-1}(A) \subset\subset f^{-1}(B)$, and thus $f^{-1}(A) \subset\subset St(f^{-1}(A),V_N) \subseteq f^{-1}(B)$ for some N. It follows as before that $A \subset\subset St(A,f(V_N)) \subseteq B$. Now, if $U_i = f(V_i)$, the conditions of corollary 8.7 are satisfied, so (Y,δ') is semi-metrizable.

Remarks. It might seem that the techniques Michael develops in [28] would be useful in showing that every semi-metrizable proximity space is the regular image of a metrizable proximity space, thus giving a negative answer to Poljakov's problem. However, if the map Michael constructs were a p-map, it is not hard to show that it would also be p-open, and the p-open image of a metrizable proximity space is metrizable [38].

8.9 Proposition. Let $f : (X,\delta) \to (Y,\delta')$ be regular and let $f^{-1}(y)$ be compact for all $y \in Y$. Then f is a (topological) quotient map.

Proof. Since f is continuous, $\zeta(\delta') \subseteq \alpha$, where α is the quotient topology on Y. Now, let U be α - open; that is, let $f^{-1}(U)$ be open. If $y \in U$, then $y \subset\subset U$ iff $f^{-1}(y) \subset\subset f^{-1}(U)$, since f is regular. But $f^{-1}(y)$ is compact and is contained in the open set $f^{-1}(U)$, so for each $x \in f^{-1}(y)$, x has an open neighborhood U_x such that $x \subset\subset U_x \subset\subset f^{-1}(U)$. Cover $f^{-1}(y)$ with a finite number of such open sets $U_{x_1},\cdots,U_{x_n}$ and let $W = \bigcup_{i=1} U_{x_i}$. Then $f^{-1}(y) \subseteq W \subset\subset f^{-1}(U)$, so that $y \in f(W) \subset\subset U$. The result follows.

Remark. Poljakov [38] gives an example of a regular map which is not a quotient map, so the condition "$f^{-1}(y)$ compact" cannot be eliminated, in

8.9. He also states that a regular, perfect (= closed with compact point-inverses) map is hereditarily quotient. If the domain has the elementary proximity or is metrizable, a slightly stronger result is true.

8.10 Proposition. Let $f : (X,\delta) \to (Y,\delta')$ be a regular map onto a separated proximity space. If either

(a) $f^{-1}(y)$ is compact for all $y \in Y$ and (X,δ) is metrizable,

OR

(b) δ is the elementary proximity and the quotient topology is completely regular,

THEN f is hereditarily quotient.

Proof. Assume (a) holds. Then by proposition 8.6, f is a quotient map. By a result of Archangel'skii [4, Theorem 1], Y is Fréchet iff f is hereditarily quotient. Clearly, a metrizable space is Fréchet. (A space X is Fréchet iff for any $E \subseteq X$, $x \in \overline{E}$ iff there is a sequence $\{x_n\} \subseteq E$ such that $x_n \to x$.)

If (b) holds and $f^{-1}(x) \subset\subset U$, where U is open, then $f^{-1}(x) \subset\subset U$. Since f is regular, $x \subset\subset f(U)$. It follows from remark 8.3(a) that f is hereditarily quotient.

Question. Under the conditions of 8.10(a) it is easy to show that $(Y,\zeta(\delta'))$ is metrizable. Is (Y,δ') metrizable?

9. Products of p-quotient Maps.

9.1 Introduction. Michael [28], Hajek [17], Siwiec [42] and others have recently considered products of quotient maps on topological spaces. It is well-known that the product of quotient maps is not, in general, a quotient map. In fact, if f is a quotient map and i_z is the identity on a paracompact Hausdorff space Z, it may not be true that $f \times i_z$ is a quotient map. Michael [26] has called quotient maps f such that $f \times i_z$ is a quotient map for any space Z, bi-quotient maps. In this section it will be shown that products of p-quotient maps behave better than products of (topological) quotient maps and we will use our result on proximities to obtain a theorem about topological quotients.

Throughout, X^* will denote the Smirnov Compactification of a proximity space (X,δ) and $h^* : X^* \to Y^*$ the unique extension of a p-map h which maps X to Y. (This extension exists by theorem 2.10). For any space Z, i_z will denote the identity map on Z.

9.2 Definition. A map (p-map) f of X onto Y is bi-quotient (p-bi-quotient) iff for every topological space (separated proximity space) Z, $f \times i_z : X \times Z \to Y \times Z$ is a quotient (p-quotient) map.

Before proving the main result of this section, we shall need two lemmas.

9.3 Lemma. Let $\{f_\alpha\}$ be a collection of p-maps such that $f_\alpha : X_\alpha \to Y_\alpha$ for all α. Then $f : \Pi X_\alpha \to \Pi Y_\alpha$ defined by $[f(x)]_\alpha = f_\alpha(x_\alpha)$ is a p-map.

Proof. It is sufficient to show $\pi_\alpha \circ f$ is a p-map for all α. But $(\pi_\alpha \circ f)(x) = f_\alpha(x_\alpha) = f_\alpha \circ \pi_\alpha(x)$, and the result follows.

9.4 Lemma. Let $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ be a p-quotient maps and let $Y_1 \times Y_2$ have the quotient proximity induced by $F = f_1 \times f_2$. Then if $x_1 \in X_1$ and $x_2 \in X_2$, $F|_{X_1 \times \{x_2\}} : X_1 \times \{x_2\} \to Y_1 \times \{f_2(x_2)\}$ and $F|_{\{x_1\} \times X_2} : \{x_1\} \times X_2 \to \{f_1(x_1)\} \times Y_2$ are p-quotient maps.

Proof. Let δ' be the quotient proximity on $Y_1 \times Y_2$, δ_s the restriction of δ' to $S = Y_2 \times \{f_2(x_2)\} = Y_2 \times \{y_2\}$, and δ_s' the quotient proximity on S induced by $F|_{X_1 \times \{x_2\}}$. Then since the restriction of a p-map is a p-map, $\delta_s < \delta_s'$. To show $\delta_s' < \delta_s$, let $A \not\delta_s' B$, where $A, B \subseteq S$. We must prove that $A \not\delta_s B$. By the definition of the quotient proximity δ_s' on S, there is some $g_s : S \to I$ such that $g_s(A) = 0$, $g_s(B) = 1$ and $G_s = g_s o(F|_{X_1 \times \{x_2\}})$ is a p-map.

Extend G_s to $X_1 \times X_2$ as follows: $G(x,y) = G_s(x,x_2)$. Further, let $g : Y_1 \times Y_2 \to I$ be defined by $g(u,z) = g_s(u,y_2)$. Now $(g \circ F)(x,y) = G(x,y)$ since $(g \circ F)(x,y) = g(F(x,y)) = g(f(x),f_2(y) = g_s(f(x),y_2) = g_s\ (F|_{X_1 \times \{x_2\}})$ $(x,x_2) = G_s(x,x_2) = G(x,y)$. Since X_1 is p-isomorphic to $X_1 \times \{x_2\}$, say by $\psi(x) = (x,x_2)$, G is equal to $G_s \circ \psi \circ \pi_{X_1}$, and thus is a p-map. Therefore, $g(A) = 0$, $g(B) = 1$ and $g \circ F = G$ is a p-map, so by the definition of the quotient proximity on $Y_1 \times Y_2$, $A \not\delta' B$. But since δ_s is the restriction of δ' to S, $A \not\delta_s B$. Hence, $\delta_s = \delta_s'$.

The corresponding result with $x_1 \in X_1$ follows similarly.

Note that this result implies that $Y_1 \times \{y_2\}$ as a subspace of $(Y_1 \times Y_2, \delta')$ is p-isomorphic to Y_1 .

9.5 Theorem. Let $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ be p-quotient maps between separated proximity spaces. Then $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is a p-quotient map.

Proof. Let δ and P be the product proximities on $X_1 \times X_2$ and $Y_1 \times Y_2$ respectively and let δ' be the quotient proximity on $Y_1 \times Y_2$ induced by $F = f_1 \times f_2$. By lemma 9.3 and the definition of the quotient proximity, $P < \delta'$. If $P \neq \delta'$, $Y_1 \times Y_2$ contains two subsets A and B such that APB but $A \not\delta' B$.

Since APB , there is a point in $Cl_{(Y_1 \times Y_2)^*} A \cap Cl_{(Y_1 \times Y_2)^*} B$, where $(Y_1 \times Y_2)^*$ is the Smirnov Compactification of $(Y_1 \times Y_2, P)$. By a result of Leader [22], $(X_1 \times X_2)^* = X_1^* \times X_2^*$ and $(Y_1 \times Y_2)^* = Y_1^* \times Y_2^*$, so say (y^*, z^*) is the point in the intersection of $Cl_{(Y_1 \times Y_2)^*} A$ and $Cl_{(Y_1 \times Y_2)^*} B$. If f_i^* , $i = 1,2$ is the extension of f_i to X_i^* , then $F^* = f_1^* \times f_2^*$ must be the unique extension of $F : (X_1 \times X_2, \delta) \to (Y_1 \times Y_2, P)$ to $X_1^* \times X_2^*$.

Consider $W = F^{*-1}(y^*, z^*) = f_1^{*-1}(y^*) \times f_2^{*-1}(z^*)$.

(i) We claim that $Cl_{(X_1 \times X_2)^*} F^{-1}(A) \cap W = \phi$.

Assume not. Then there is an open set U containing W such that $U \cap F^{-1}(A) = \phi$. But since $f_1^{*-1}(y^*)$ and $f_2^{*-1}(z^*)$ are compact, open sets U_1 in X_1^* and U_2 in X_2^* exist such that $W \subseteq U_1 \times U_2 \subseteq U$. Now,

f_1^* and f_2^* are closed maps, hence easily hereditarily quotient. It follows from remark 8.3 (a) that $f_1^*(U_1)$ is a neighborhood of y^* and $f_2^*(U_2)$ is a neighborhood of z^* ; hence,
$F^*(U_1 \times U_2) = f_1^*(U_1) \times f_2^*(U_2)$ is a neighborhood of (y^*, z^*) . But $U_1 \times U_2 \subseteq U$ and $F^*(U) \cap A = \phi$ - which contradicts the fact that $(y^*, z^*) \in Cl_{(Y_1 \times Y_2)^*} A$. <u>This establishes the claim.</u> Similarly, $Cl_{(X_1 \times X_2)^*} F^{-1}(B) \cap W \neq \phi$.

Let (a,b) be a point in $Cl_{(X_1 \times X_2)^*} F^{-1}(A) \cap W$ and let (c,d) be a point in $Cl_{(X_1 \times X_2)^*} F^{-1}(B) \cap W$.

Since $A \not\delta' B$, there is a function $g : Y \to I$ such that $g(A) = 0$, $g(B) = 1$ and $g \circ F$ is a p-map. Our objective is to show that the extension, $(g \circ F)^*$, of $g \circ F$ to $(X_1 \times X_2)^*$ takes (a,b) and (c,d) to the same point in I . But first, <u>let $x_2 \in X_2$</u> .

(ii) <u>We claim that</u> $(g \circ F)^*(a, x_2) = (g \circ F)^*(c, x_2)$.

If we consider F as a map onto $(Y_1 \times Y_2, \delta')$, then by lemma 9.4, $Y_2 \times \{y_2\} = Y_2 \times \{f_2(x_2)\}$ has the quotient proximity induced by $F|_{X_1 \times \{x_2\}}$. Then, since $(g \circ F)|_{X_1 \times \{x_2\}} = (g|_{Y_1 \times \{y_2\}}) \circ (F|_{X_1 \times \{x_2\}})$, it follows from theorem 4.7 that $g|_{Y_1 \times \{y_2\}}$ is a p-map. But the extension of a p-map to the Smirnov Compacification is unique, hence

(iii) $((g \circ F)|_{X_1 \times \{x_2\}})^* = (g \circ F)^*|_{X_1^* \times \{x_2\}} = (g|_{Y_1 \times \{y_2\}})^* \circ (F|_{X_1 \times \{x_2\}})^*$

where $((g \circ F)|_{X_1 \times \{x_2\}})^*$ is the extension of $(g \circ F)|_{X_1 \times \{x_2\}}$ to

$X_1^* \times \{x_2\}$ and $(g|_{Y_1\times\{y_2\}})^*$ is the extension of the p-map $g|_{Y_1\times\{y_2\}}$ to $(Y_1\times\{y_2\})^*$ (which is $Y_1^*\times\{y_2\}$ by the remark following lemma 9.4). Now, $f_1^*(a) = f_1^*(c) = y^*$, so $(F|_{X_1\times\{x_2\}})^*(a,x_2) = (F|_{X_1\times\{x_2\}})^*(c,x_2)$ and hence $(g|_{Y_1\times\{y_2\}})^* \circ (F|_{X_1\times\{x_2\}})^*(a,x_2) = (g|_{Y_1\times\{y_2\}})^* \circ (F|_{X_1\times\{x_2\}})^*(c,x_2)$. It follows from equation (iii) above that $(g\circ F)^*(a,x_2) = (g\circ F)^*(c,x_2)$, establishing claim (ii). If we repeat the above argument with the roles of X_1 and X_2 interchanged, then it follows similarly that $(g\circ F)^*(x_1,b) = (g\circ F)^*(x_1,d)$ for any $x_1 \in X_1$.

(iv) We now use a limiting process to show $(g\circ F)^*(a,b) = (g\circ F)^*(c,d)$. Pick a net $\langle(a_\alpha,b_\alpha)\rangle$ in $F^{-1}(A)$ converging to (a,b). Then $\langle b_\alpha\rangle \subseteq X_2$ converges to b, so $\langle(a,b_\alpha)\rangle \to (a,b)$ and $\langle(c,b_\alpha)\rangle \to (c,b)$. But for each α, it follows from (ii) that $(g\circ F)^*(a,b_\alpha) = (g\circ F)^*(c,b_\alpha)$, hence in the limit, $(g\circ F)^*(a,b) = (g\circ F)^*(c,d)$. If we pick a net in $F^{-1}(B)$ converging to (c,d), a similar argument will show $(g\circ F)^*(a,d) = (g\circ F)^*(c,d)$. Again, we can easily interchange the roles of X_1 and X_2 to show that $(g\circ F)^*(a,d) = (g\circ F)^*(a,b)$ and $(g\circ F)^*(c,d) = (g\circ F)^*(c,b)$. Putting these together we easily verify (iv).

However, since $g(A) = 0$, $(g\circ F)(F^{-1}(A)) = 0$, so that $(g\circ F)^*(\mathrm{Cl}_{(X_1\times X_2)^*} F^{-1}(A)) = 0$. Similarly $(g\circ F)^*(\mathrm{Cl}_{(X_1\times X_2)^*} F^{-1}(B)) = 1$. In particular, $(g\circ F)^*(a,b) = 0$ and $(g\circ F)^*(c,d) = 1$ – which contradicts (iv).

It follows that $\delta' = P$ and $f_1\times f_2$ is a p-quotient map. This completes the proof.

9.5 Corollary. If $f_i : X_i \to Y_i$, $i = 1,\cdots,n$ are p-quotient maps between separated proximity spaces, then $F : \prod_{i=1}^{n} X_i \to \prod_{i=1}^{n} Y_i$ defined by $F(x_1,\cdots,x_n) = (f_1(x) , \cdots , f_n(x_n))$ is a p-quotient map.

9.6 Corollary. A p-quotient map between separated proximity spaces is p-bi-quotient.

Before proceeding to an application of theorem 9.4, we need a lemma which is of some independent interest. Hager in [16] has announced a similar result for uniform spaces.

9.7 Definition. A topological space X is pseudocompact iff every real-valued continuous function on X is bounded.

9.8 Lemma. Let X and Y be infinite separated proximity spaces each with the fine proximity. Then the product proximity is the fine proximity on $X\times Y$ iff $X\times Y$ is pseudocompact.

Proof. Since there is a one-to-one order preserving correspondence between proximities and compactifications, $(X\times Y \text{ , product proximity})^* < (X\times Y \text{ , fine proximity})^*$. By Leader's result [22, theorem 10] $(X\times Y)^* = X^*\times Y^*$ and since the Smirnov Compactification of a space with the fine proximity is just the Stone-Cech Compactification, we have

$$BX\times BY = X^*\times Y^* = (X\times Y \text{ , product})^* < (X\times Y \text{ , fine})^* = B(X\times Y) \quad .$$

Now, Glicksberg [15] has shown $BX\times BY = B(X\times Y)$ iff $X\times Y$ is pseudocompact. The result follows.

<u>9.9 Theorem</u>. Let $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ be (topological) quotient maps between $T_{3\frac{1}{2}}$ spaces. Then if $X_1 \times X_2$ is pseudocompact, $f_1 \times f_2$ is a quotient map.

<u>Proof</u>. Let X_1, X_2, Y_1 and Y_2 each have the fine proximity. Then by lemma 9.8, the product proximity on $X_1 \times X_2$ is the fine proximity and since $Y_1 \times Y_2$ is $T_{3\frac{1}{2}}$, it follows from theorem 5.3 that the quotient topology is equal to $\zeta(\delta')$. Now, by theorem 9.4, δ' is the product proximity P on $Y_1 \times Y_2$ so $\zeta(P) = \zeta(\delta')$ is equal to the quotient topology and hence, $f \times g$ is a quotient map.

10. <u>Examples and Relationships</u>.

In this section we explore the relationships between the maps introduced in sections 8 and 9 and their topological equivalents.

<u>10.1 Example</u>. A bi-quotient, p-quotient map which is not regular.

Let $X = [0,2]$ and identify $[\frac{1}{2},\frac{3}{2}]$ to a point. Then the natural map is easily bi-quotient and p-quotient. But $f[0,\frac{1}{2}]\delta' f(\frac{3}{2},2]$, although $[0,\frac{1}{2}) \not\delta (\frac{3}{2},2]$.

Example 10.1 contrasts with topological quotient maps since bi-quotient maps are hereditarily quotient.

<u>10.2 Example</u>. A regular map which is not quotient.

Let $X = [0,1] \times \{0,1\}$ and let Y be formed by identifying $(x,0)$ and $(x,1)$ for all $x > 0$. Then the quotient topology is not

completely regular, so by theorem 5.3, $\zeta(\delta')$ is not the quotient topology. But it is not hard to show that $A\delta' B \iff f^{-1}(A)\delta f^{-1}(B)$.

<u>10.3 Example</u>. A hereditarily quotient, p-quotient map which is not bi-quotient.

Let X be the disjoint union of countably many copies of $[0,1]$ and let X have the elementary proximity. Identify all 0s to a common point. Then δ with the quotient topology is completely regular, so the natural map f is quotient and p-quotient. f is also easily hereditarily quotient. But Michael has shown [28, example 8.1] that f is not bi-quotient.

<u>10.4 Remark</u>. Michael [28] has shown that if Y is T_2 and $f : X \to Y$, the following are equivalent:

(a) f is bi-quotient.

(b) For $y \in Y$, if $\mathcal{U}$ is any open cover of X , then finitely many $f(U)$, $U \in \mathcal{U}$, cover some neighborhood of y .

(c) For $y \in Y$, if $\mathcal{U}$ is any open cover of $f^{-1}(y)$, then finitely many $f(U)$, $U \in \mathcal{U}$, cover some neighborhood of y .

(d) $f \times i_Z$ is p-quotient map for every paracompact space Z .

The following analogue holds for proximity spaces. The proof is similar to Michael's.

<u>10.5 Proposition</u>. If Y is a separated proximity space and $f : X \to Y$ is a p-map, then the following are equivalent:

(a) For A, a closed subset of Y, $\mathcal{U}$ a p-cover of $f^{-1}(A)$, then finitely many $f(U)$, $U \in \mathcal{U}$ cover some p-neighborhood of A.

(b) For A, a closed subset of Y, $\mathcal{U}$ a p-cover of X, then finitely many $f(U)$, $U \in \mathcal{U}$, cover some p-neighborhood of A.

(A cover $U = \{U_\alpha\}_{\alpha\in A}$ is a <u>p-cover</u> of X iff there is a cover $V = \{V_\alpha\}_{\alpha\in A}$ such that for all α $V_\alpha \subset\subset U_\alpha$.)

<u>Proof</u>. $a \Rightarrow b$ is clear. To show $b \Rightarrow a$, let A be a closed subset of Y and $\mathcal{U}$ a p-cover of $f^{-1}(A)$. Since Y is T_2, for each $x \notin A$ pick W_x and V_x such that $W_x \subset\subset V_x$ and $V_x \not\subset A$. Let $\mathcal{V} = \{V_x\}_{x\notin A}$ and $\mathcal{W} = \mathcal{U} \cup f^{-1}(\mathcal{V})$. Then $\mathcal{W}$ is a p-cover of X, so there is a set $N = C \cup D$ such that $A \subset\subset N$, where C is the union of finitely many $f(U)$, $U \in \mathcal{U}$ and D is the union of finitely many V_x. But then $A \not\subset D$ implies $A \subset\subset C$.

<u>10.6 Remark</u>. Remark 10.4 might lead us to expect that 10.5(a) is equivalent to f being p-bi-quotient and hence that 10.5(a) holds for every p-quotient map. However, since 10.5(a) easily implies that f is regular, example 10.1 shows that this is not the case.

CHAPTER IV

PROXIMITY ANALOGUES OF TWO METRIZATION AND MAPPING THEOREMS

11. The Two Theorems.

In this chapter we consider proximity analogues of a metrization theorem of Morita and the Morita-Hanai-Stone Theorem on the closed continuous image of a metric space. Some of our results on proximity spaces will be stated for the wider class of generalized proximity spaces introduced by Lodato in [25] and [26]. We reproduce his definition here.

11.1 Definition. A Lodato proximity (or LO-proximity) space is a pair (X,δ) where X is a set and δ a binary relation on $\mathcal{P}(X)$ which satisfies:

P1) if $A\delta B$, then $A \neq \phi$ and $B \neq \phi$,

P2) $A\delta B$ iff $B\delta A$,

P3) if $A \cap B \neq \phi$, then $A\delta B$,

P4) $A\delta(B \cup C)$ iff $A\delta B$ or $A\delta C$,

L) $A\delta B$ and $b\delta C$ for all $b \in B$, imply $A\delta C$.

As with proximity spaces, a LO-proximity is separated if it also satisfies:

P6) $a = b$ iff $a\delta b$.

All LO-proximities and proximities in the chapter will be assumed separated. Perhaps it should be pointed out that lemma 8.6 of the last chapter

could have been stated for LO-proximities.

In [30], Morita proved that a T_1 - space X is metrizable iff there is a sequence $\{F_i\}_{i=1}^{\infty}$ of locally finite closed covers of X such that whenever U is an open neighborhood of $x \in X$, there is some F_N with $St(x,F_N) \subseteq U$. This result has an analogue for proximity spaces (or LO-proximity spaces) with the elementary proximity, as theorem 11.3 demonstrates. We will need to make use of the following theorem of Smirnov [43, Theorem 16'].

<u>11.2 Theorem.</u> A LO-proximity space (X,δ) is metrizable iff there is a sequence $\{U_i\}_{i=1}^{\infty}$ of covers of X such that U_{i+1} star-refines U_i, $A \subset\subset St(A,U_i)$ for all i, and whenever $A \subset\subset B$, there is some U_N which satisfies $A \subset\subset St(A,U_N) \subset B$.

<u>11.3 Theorem.</u> Let (X,δ) be a LO-proximity space with $A\delta B$ iff $\overline{A} \cap \overline{B} \neq \phi$. Then (X,δ) is metrizable iff there is a sequence $\{F_i\}_{i=1}^{\infty}$ of locally finite closed covers of X such that whenever A is closed and $A \subset\subset B$, there is some F_N which satisfies $A \subseteq St(A,F_N) \subseteq B$.

<u>Proof.</u> Let (X,δ) be metrizable. Then by the metrization theorem of Smirnov there is a sequence $\{U_i\}_{i=1}^{\infty}$ of open covers of X such that U_{i+1} star refines U_i, $A \subset\subset St(A,U_i)$ for all i, and whenever $A \subset\subset B$, there is some U_N which satisfies $A \subset\subset St(A,U_N) \subset B$. Since metrizable spaces are paracompact, each U_i has a locally finite closed refinement F_i. Clearly, $\{F_i\}_{i=1}^{\infty}$ has the required properties.

Conversely, given such a collection $\{F_i\}_{i=1}^{\infty}$ let $\overline{M}_i = \{\overline{M(\alpha,i)} \mid \alpha \in \Omega_i\}$ be a "grating" for F_i for each i. That is, $\overline{M}_i < F_i$, $M(\alpha,i)$ is open, $\overline{M}_i$ is a locally finite closed cover, and $M(\alpha,i) \cap M(B,i) = \phi$ if $\alpha \neq B$. This grating exists by Lemma 3 of Morita's paper [30]. For a given i, define $W(\alpha_1 \cdots \alpha_i) = \bigcap_{j=1}^{i} M(\alpha_j,j)$ where $\overline{M(\alpha_j,j)} \in \overline{M}_j$ and let $\overline{W}_i = \{\overline{W(\alpha_1 \cdots \alpha_i)} \mid \alpha_j \in \Omega_j, j = 1,2,\cdots,i\}$. Then by lemma 4 of [30], $\overline{W}_i$ is also grating. Clearly, $\overline{W}_i < \overline{M}_i < F_i$ and $\overline{W}_i < \overline{W}_{i-1}$.

Now, for a closed subset A of X, let $V_n(A) = \operatorname{int}\{St(A,\overline{W}_n)\}$. Note that $A \subseteq V_n(A)$ since $A \subseteq C = \{X - \cup\{\overline{W(\alpha_1 \cdots \alpha_n)} \mid A \cap \overline{W(\alpha_1 \cdots \alpha_n)} = \phi\} \subseteq St(A,\overline{W}_n)$ and C is open because $\overline{W}_n$ is locally finite. Since $A \subset\subset V_n(A)$, for each n there is a $m = m(A,n) > n$ such that $A \subseteq St(A,\overline{W}_m) \subseteq V_n(A)$.

Let A be a closed subset of X and $y \in X$. We claim that if $V_m(y) \cap V_m(A) \neq \phi$, then $V_m(y) \subseteq V_n(A)$.

(i) $y \in St(A,\overline{W}_m)$. If not, then whenever $A \cap \overline{W(\alpha_1 \cdots \alpha_m)} \neq \phi$ and $y \in \overline{W(\beta_1 \cdots \beta_m)}$, we have $W(\alpha_1 \cdots \alpha_m) \cap W(\beta_1 \cdots \beta_m) = \phi$, since $\overline{W}_m$ is a grating. But then $\overline{W(\alpha_1 \cdots \alpha_m)} \cap W(\beta_1 \cdots \beta_m) = \phi$ and $V_m(A) \cap V_m(y) = \phi$ – a contradiction.

(ii) Let $y \in \overline{W(\beta_1 \cdots \beta_m)}$. Then we claim $A \cap \overline{W(\beta_1 \cdots \beta_n)} \neq \phi$. For if $A \cap \overline{W(\beta_1 \cdots \beta_n)} = \phi$, $W(\alpha_1 \cdots \alpha_n) \cap W(\beta_1 \cdots \beta_n) = \phi$ for any $\overline{W(\alpha_1 \cdots \alpha_n)}$ such that $A \cap W(\alpha_1 \cdots \alpha_n) \neq \phi$. Hence, as before $V_n(A) \cap \overline{W(\beta_1 \cdots \beta_n)} = \phi$. But $y \in St(A,\overline{W}_m) \cap \overline{W(\beta_1 \cdots \beta_m)} \subseteq V_n(A) \cap \overline{W(\beta_1 \cdots \beta_n)}$ – a contradiction.

(iii) $St(y,\overline{W}_m) \subseteq St(A,\overline{W}_n)$ follows from (i) and (ii). Thus we have $V_m(y) \subseteq V_n(A)$, establishing the claim.

For each closed subset A of X , let $n_1(A) = 1$, $n_r(A) = m(A,n_{r-1}(A))$ and let $U_r(A) = V_{n_r}(A)$. Consider $\mathcal{U}_r = \{U_r(x) | x \in X\}$.

If $U_r(y) \cap U_r(A) \neq \phi$, $V_{n_r}(y) \cap V_{n_r}(A) \neq \phi$ so $U_r(y) \subseteq U_{r-1}(A)$. It follows that $St(U_r(A),\mathcal{U}_r) \subseteq U_{r-1}(A)$, and therefore $St(U_r(x),\mathcal{U}_r) \subseteq U_{r-1}(x)$ and $\mathcal{U}_r \overset{*}{<} \mathcal{U}_{r-1}$. Now let $A \subset\subset U$. Then there is an r such that $A \subseteq St(A,\mathcal{U}_{r+1}) \subseteq U_r(A) \subset V_{n_r}(A) \subseteq V_r(A) \subseteq U$.

Thus, by Smirnov's Theorem, (X,δ) is metrizable. This completes the proof.

<u>Remark</u>. The hypothesis that the LO-proximity be "elementary" in the above theorem is certainly not needed in the proof of necessity. I do not know whether it can be eliminated in proving sufficiency.

Our next result will be a proximity analogue of the theorem of Morita and Hanai [31] and Stone [45] on closed mappings of metric spaces. The proof of the following lemma is contained in the proof of theorem 1 of [31].

<u>11.4 Lemma</u>. Let X and Y be metrizable topological spaces and f a closed continuous map of X onto Y . Then there is a closed subset X_o of X such that $f|_{X_o}^{-1}(y)$ is compact for every $y \in Y$. Further, if $\{F_i\}_{i=1}^{\infty}$ is a sequence of locally finite closed covers of X , then $\{f(F_i)\}_{i=1}^{\infty}$ is a sequence of locally finite closed covers of Y .

This completes the claim.

Let $\{V_i\}_{i=1}^{\infty}$ be an open neighborhood base for the set A. Now $x_i \in \partial f^{-1}(A)$ and $U_i \cap f^{-1}(V_i)$ is a neighborhood of x_i, so for each i there is some $x_i' \in (U_i \cap f^{-1}(V_i)) - f^{-1}(A)$. Since $C = \{x_i'\}$ is locally finite, it is closed, hence $f(C)$ is closed. If $H = Y - f(C)$, then $A \subset\subset H$ since A and $f(C)$ are disjoint closed sets. It follows that there is some V_i with $A \subset\subset V_i \subset H$. But $x_i' \in f^{-1}(V_i)$, so $f(x_i') \in V_i \subseteq Y - f(C)$ - a contradiction.

<u>11.5 Corollary</u>. Let X be a normal T_1 space such that every closed set has a countable neighborhood base. Then ∂A is countably compact for every closed subset A of X.

Willard [50] has called a metrizable space whose set of accumulation points is compact, an <u>A-space</u>. These spaces have also been studied by Rainwater [41] and MacDonald and Willard [27].

11.5 Lemma. Let (X,δ) be a proximity space whose induced topology is normal and (Y,δ') a LO-proximity space with $A\delta'B$ iff $\overline{A}\cap\overline{B} \neq \phi$. Let every closed subset of Y have a countable neighborhood base. Then if f is a closed p-map of X onto Y, the boundary, $\partial f^{-1}(A)$, of $f^{-1}(A)$ is countably compact for every closed subset A of Y.

Proof. Suppose $\partial f^{-1}(A)$ is not countably compact for some closed subset A of Y. Then there exists a sequence $\{x_i\} \subseteq \partial f^{-1}(A)$ without a cluster point in $\partial f^{-1}(A)$ and hence without a cluster point in X.

We claim there is a locally finite collection $\{U_i\}_{i=1}^{\infty}$ of disjoint open sets in X with $x_i \in U_i$ for all i. To see this, first use normality to find a collection $\{W_i'\}_{i=1}^{\infty}$ of disjoint open sets with $x_i \in W_i'$ for all i. Now, for each i pick an open set W_i such that $x_i \in W_i \subseteq \overline{W}_i \subseteq W_i'$. Consider $Z = \overline{\cup W_i} - \cup\overline{W}_i$. If $Z = \phi$, let $U_i = W_i$. If not, for each $x \in Z$ there is a neighborhood $U(x)$ such that $U(x) \cap \{x_i\} = \phi$, since x is not a limit point of $\{x_i\}$. Let $U = \bigcup_{x\in Z} U(x)$. Now, $Z \subseteq U$ and X is normal so there is an open set V such that $Z \subseteq V \subseteq \overline{V} \subseteq U$. Let $W = X - \overline{V}$ and let $U_i = W \cap V_i$. Clearly, $x_i \in U_i$ for each i. $\{U_i\}_{i=1}^{\infty}$ is the desired collection; for if $x \in X$ then:

Case 1: If $x \in \overline{\cup W_i} - \cup\overline{W}_i$, $x \in V$. But $V \cap \cup U_i = \phi$, so V is a neighborhood of x intersecting no U_i.

Case 2: If $x \notin \overline{\cup W_i}$, x has a neighborhood intersecting no U_i.

Case 3: If $x \in \cup\overline{W}_i$, say $x \in \overline{W}_k$, then $x \in W_k'$, which is a neighborhood of X intersecting U_k and no other U_i.

<u>11.6 Theorem</u>. Let (X,δ) be a metrizable proximity space, (Y,δ') a LO-proximity space where $A\delta' B$ iff $\overline{A}\cap\overline{B} \neq \phi$ and $f : X \to Y$ a closed p-map onto Y. Then

(a) (Y,δ') is metrizable,

(b) $\partial f^{-1}(A)$ is compact for every closed subset A of Y,

(c) every closed subset A of Y has a countable neighborhood base,

(d) Y is an A - space,

(e) X has a closed subspace Z which is an A-space and which maps onto Y,

(f) Y is complete.

<u>Proof</u>. First, we show Y is metrizable. Let $y \in Y$. Since (X,δ) is metrizable, the metrization theorem of Smirnov implies there is a collection $\{U_i\}_{i=1}^{\infty}$ of open sets in X such that $f^{-1}(y) \subset\subset U_i$ for all i and whenever $f^{-1}(y) \subset\subset U$, there is some U_N which satisfies $f^{-1}(y) \subset\subset U_N \subset U$. If $V_i = Y - f(X-U_i)$, it is routine to show $\{V_i\}_{i=1}^{\infty}$ is a neigborhood base for y. It follows from the theorem of Hanai and Morita [31] and Stone [45] that Y is metrizable (as a topological space).

By the remark following Theorem 11.3, there is a sequence $\{\mathcal{F}_i\}_{i=1}^{\infty}$ of locally finite closed covers of X such that whenever A is a closed subset of X with $A \subset\subset B$, there is some $\mathcal{F}_N$ which satisfies $A \subseteq St(A,\mathcal{F}_N) \subseteq B$. To prove that (Y,δ') is metrizable, it is sufficient

by theorem 11.3 and lemma 11.4 to demonstrate that $\{f(F_i)\}_{i=1}^{\infty}$ has the required properties. So let A be a closed subset of Y with $A \subset\subset B$. Then $f^{-1}(A)$ is closed and $f^{-1}(A) \subset\subset f^{-1}(B)$ since f is a p-map; hence, there is some F_N which satisfies $f^{-1}(A) \subseteq St(f^{-1}(A), F_N) \subseteq f^{-1}(B)$. But then (easily) $A \subseteq St(A, f(F_N)) \subseteq B$.

Rainwater [41] has shown (a), (c) and (d) are equivalent; so (c) and (d) hold. (b) follows from (c) by lemma 11.5. It remains to show (e) and (f). Since Y is an A-space, let $Y = Y_o \cup D$, where Y_o is the (compact) set of accumulation points. By lemma 11.4 there is a closed subset X_o of X such that $f(X_o) = Y$, $f|_{X_o}$ is a closed p-map and $f|_{X_o}^{-1}(y)$ is compact for all $y \in Y$. Let $X_1 = f^{-1}(Y_o) \cap X_o$ and for each $y \in D$, pick some $x \in f^{-1}(y) \cap X_o$, and let X_2 be the set of all x so chosen. Then X_1 is compact since $f|_{X_o}$ is perfect. Let $Z = X_1 \cup X_2$. If $x_n \in Z$ and $x_n \to x_o \in X_o$, then $f(x_n) \to f(x_o)$ so $f(x_o) \in Y_o$. But then $x_o \in f^{-1}(Y_o) \cap X_o = X_1 \subseteq Z$. Therefore Z is closed in X_o, and hence in X. That the set of accumulation points of Z is contained in X_1, and hence compact, follows similarly.

Since every A-space is complete [50], (f) holds.

CHAPTER V

PRODUCTS OF PROXIMITY SPACES

12. Elementary Products.

12.1 Introduction. Although the usual product proximity generates the product topology and is the correct categorical product, it is in many ways inadequate. As is well-known, the product proximity on R^2 is not the usual metric proximity. In fact, the lines $y = x$ and $y = x+5$ are "near" in the product proximity. Poljakov [40] has defined another product which does have the property that the product of metrizable proximity spaces is metrizable. The purpose of the product presented here is to give the elementary proximity on the product in terms of factors.

12.2 Definition. Let $\{(X_\alpha, \delta_\alpha)\}$ be a collection of sets and binary relations each satisfying (P1)-(P4) of definition 2.1 and define $A \delta B$ in $X = \prod X_\alpha$ iff there is an $x \in X$ such that whenever $A = A_1 \cup \cdots \cup A_n$ and $B = B_1 \cup \cdots \cup B_m$, then for some A_i and B_j, $\pi_\alpha(x) \delta_\alpha \pi_\alpha(B_j)$ and $\pi_\alpha(A_i) \delta_\alpha \pi_\alpha(x)$ for all α.

12.3 Lemma. (X, δ) satisfies (P1)-(P4) of definition 2.1 and each projection $\pi_\alpha : X \to X$ is a p-map.

Proof. (P1)-(P3) are easy. Let $A \not\delta B$ and $A \not\delta C$. We shall show $A \not\delta (B \cup C)$. Assume $A \delta (B \cup C)$. Then there is some $x \in \prod X_\alpha$ such that whenever $A = A_1 \cup \cdots \cup A_m$ and $(B \cup C) = D_1 \cup \cdots \cup D_k$, then for some A_i and D_j

$\pi_\alpha(x)\delta_\alpha\pi_\alpha(A_i)$ and $\pi_\alpha(x)\delta_\alpha\pi_\alpha(D_j)$ for all α . But since $A\not\delta B$ and $A\not\delta C$, there are decompositions of B and C , say $B = B_1\cup\cdots\cup B_N$ and $C = C_1\cup\cdots\cup C_M$ such that for all B_i , $i = 1,\cdots,N$ there exists an α_i such that $\pi_{\alpha_i}(B_i)\not\delta_{\alpha_i}\pi_{\alpha_i}(x)$ and for all C_i , $i = 1,\cdots,M$ there exists an α_i such that $\pi_{\alpha_i}(C_i)\not\delta_{\alpha_i}\pi_{\alpha_i}(x)$. But then $B\cup C = (B_1\cup\cdots\cup B_N) \cup (C_1\cup\cdots\cup C_M)$ and for all $i = 1,\cdots,N$, $N+1$, $\cdots$, $N+M$ there exits an α_i such that $\pi_{\alpha_i}(D_i)\not\delta_{\alpha_i}\pi_{\alpha_i}(x)$, where $D_i = B_i$, $i = 1,\cdots,N$. $D_i = D_{N+j} = B_j$, $j = 1,\cdots,M$ - a contradiction. The other implication of (P4) is easy.

Since $A\delta B$ implies that whenever $A = A_1\cup\cdots\cup A_n$ and $B = B_1\cup\cdots\cup B_n$ then for some A_i , B_j $\pi_\alpha(A_i)\delta\pi_\alpha(B_j)$ for all α , thus each π_α is a p-map.

The proof of the following lemma is easy but tedious and thus omitted.

12.4 Lemma. Let $\{(X_\alpha,\delta_\alpha)\}$ be as in 12.2 and let $kA = \{x|x\ \delta\ A\}$. If each (X_α,δ_α) also satisfies: $x\delta_\alpha A$ iff $x\delta_\alpha kA$ where $A \subseteq X_\alpha$, and U is a basic open set in the product topology, then U is open in the topology generated by any generalized proximity on X for which the projections are p-maps.

12.5 Remark. $x\delta A$ iff whenever $A = A_1\cup\cdots\cup A_n$, then there is some A_i such that $\pi_\alpha(x)\delta_\alpha\pi_\alpha(A_i)$ for all α .

12.6 Theorem. If $\{(X_\alpha, \delta_\alpha)\}$ is a collection of LO-proximities then δ generates the product topology.

Proof. It follows from the definition of δ and the remark above that $x \delta A$ iff $x \delta kA$, where kA is as in lemma 12.4. It remains to show that if U is open in the induced topology, then U is open in the product topology.

Let $p \in U$. So $p \not\delta (X-U)$. It follows that there is a decomposition of $X-U$, say $X-U = U_1 \cup \cdots \cup U_n$ such that for each U_K there is some α_K with $\pi_{\alpha_K}(p) \not\delta \alpha_K \pi_\alpha(U_K)$. For $K = 1,2,\cdots,n$, let $W_K = X_{\alpha_K} - \overline{\pi_{\alpha_K}(U_K)}$. Since $\pi_{\alpha_K}(p) \not\delta \alpha_K \pi_{\alpha_K}(U_K)$, $\pi_{\alpha_K}(p) \not\delta_{\alpha_K} \overline{\pi_{\alpha_K}(U_K)}$ and hence $\pi_{\alpha_K}(p) \in W_K$. We claim $p \in \bigcap_{K=1}^{n} \pi_{\alpha_K}^{-1}(W_K) \subseteq U = \bigcup_{K=1}^{n} (X-U_K))$. But $\pi_{\alpha_K}(x) \in W_K = X_{\alpha_k} - \overline{\pi_{\alpha_K}(U_k)}$, and thus, $\pi_{\alpha_K}(x) \notin \pi_{\alpha_K}(U_K)$. The result follows.

12.7 Corollary. If $\{(X_\alpha, \delta_\alpha)\}$ is a collection of LO-proximities then (X,δ) as defined in 12.2 is a LO-proximity and $A \delta B$ iff $\overline{A} \cap \overline{B} \neq \phi$, where closure is in the product topology.

Proof. Easily $A\delta B$ if $\overline{A} \cap \overline{B} \neq \phi$. Also, given (X,δ) satisfying (P1)-(P4) of 2.1, (X,δ) is a LO-proximity exactly when $A\delta B$ iff $kA\delta kB$, where k is the operator defined in 12.4. But this clearly holds.

12.8 Corollary. If $\{(X_\alpha, \delta_\alpha)\}$ is a collection of proximity spaces and the product topology is normal, then the product defined in 12.2 is a proximity, generates the product topology, and $A\delta B$ iff $\overline{A} \cap \overline{B} \neq \phi$.

Note that if X and Y are two infinite normal proximity spaces with the elementary proximity and if $X \times Y$ is normal, then by lemma 9.8, the product proximity on $X \times Y$ is given by 12.2 iff $X \times Y$ is pseudocompact.

Dr. S. Leader has pointed out that the product proximity of this section may also be considered as a space with a pseudoclosure operator: (i) $\overline{\phi} = \phi$, (ii) $A \subseteq \overline{A}$, (iii) $\overline{A \cup B} = \overline{A} \cup \overline{B}$. Each δ satisfying (P1)-(P4) induces a pseudoclosure, namely $\overline{A} = \{x \mid x\delta A\}$. Definition 12.2 depends only on the pseudoclosures induced by the δ_α's. The product then comes from the product pseudoclosure: $A\delta B$ iff $\overline{A} \cap \overline{B} \neq \phi$.

13. Products of Separation Spaces.

We now consider products of another type of proximity space. Wallace [47] defined a strong separation space to be a pair (X,δ) where X is a set and δ is a binary relation on $P(X)$ satisfying (P1)-(P4) and (P6) of definition 2.1 and in addition

(S1) $x\delta A$ iff $x\delta kA$ where $kA = \{x | x\delta A\}$,

(S2) if $A\delta B$ then either there is an $a \in A$ such that $a\delta B$ or there is a $b \in B$ such that $b\delta A$.

Wallace showed that for T_1 spaces, this is exactly topological separation. These are the fine S-proximity spaces that Gagrat and Naimpally have recently considered ([11] and [12]). Our original motivation for this problem was to find an axiomatization for topological separation on a product of topological spaces to facilitate the study of functions with connected graphs.

13.1 Definition. Let $\{(X_\alpha,\delta_\alpha)\}$ be a collection of pairs each satisfying (P1)-(P4) of definition 2.1 and define $A\delta B$ in $X = \Pi X$, iff either there is an $a \in A$ such that whenever $B = B_1 \cup \cdots \cup B_n$ then for some B_j , $\pi_\alpha(a)\delta_\alpha\pi_\alpha(B_j)$ for all α , or there is a $b \in B$ such that whenever

$A = A_1 \cdots A_m$ then for some A_i , $\pi_\alpha(A_i)\delta_\alpha\pi_\alpha(b)$ for all α .

The proof of the following theorem follows the general outline of section 12.

<u>13.2 Theorem.</u> If $\{(X_\alpha,\delta_\alpha)\}$ are strong separation spaces then δ as defined in 13.1 is a strong separation on $X = \Pi X_\alpha$ and generates the product topology. $A\delta B$ iff $\overline{A}\cap B \neq \phi$ or $A\cap\overline{B} \neq \phi$ and if $f : (Z,\delta^1) \to (\Pi X_\alpha,\delta)$, where (Z,δ^1) is a strong separation space, then f is a p-map iff $\pi_\alpha \circ f$ is a p-map for all α .

Thus, definition 13.1 gives us an axiomatization of topological separation on any product of T_1 spaces.

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